Quantum Walks on Noncommutative Geometries

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Abstract

We study the discrete time quantum walk of a Dirac particle in 2+1 dimensions. After proving that the discretized evolution preserves relativistic causality, we explore the geometric consequences of introducing a magnetic gauge field. The gauge coupling prevents spatial shift operators from commuting, transforming the underlying lattice into a noncommutative geometry. Using Alain Connes distance, we analytically compute the distance between localized Gaussian states in the commutative/noncommutative and continuous limits. Finally, we formulate the noncommutative distance on the discrete lattice as a constrained optimization problem, demonstrating how the magnetic field fundamentally entangles spatial and momentum degrees of freedom.
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1 Introduction

Quantum walks are the quantum analogues of classical random walks. The Discrete Time Quantum Walk (DTQW) [1, 2, 3] simulates the Dirac equation, which provides a framework for studying relativistic quantum mechanics on a lattice. Discretizing the continuous time evolution of a Dirac particle gives a local unitary operator that defines the dynamics of a quantum walker. In a standard continuous space, or on a simple grid, the geometry governing this walk is commutative.

However, introducing a magnetic field changes the underlying structure of the space. In the continuous regime, a magnetic field couples to the momentum of the particle. On a discrete lattice, this gauge coupling modifies the spatial translation operators. When the magnetic field is non zero, spatial shifts along the x and y axes no longer commute ($[T_x, T_y] \neq 0$). The walker accumulates position dependent phases depending on the path taken. If we suppose that the two spatial axes do not commute anymore, we lose the classical notion of points because of Heisenberg uncertainty principle that prevents the walker from being exactly localized in x and y simultaneously. Therefore, the standard Euclidean grid breaks down, forcing us into noncommutative geometry.

Noncommutative Geometry (NCG) [4, 5, 7] reconstructs geometric properties such as distance from the algebra of observables, without relying on classical coordinate points, which is useful in order to study the spread of a quantum walker in such regimes. In this noncommutative setting, the most reasonable way to approximate the exact spatial points that we lost is through Gaussian wavepackets (rotation invariant states of minimal uncertainty). By evaluating the Connes distance [6] between such states, we can rigorously estimate how far the walker has spread in a space where position and momentum are entangled.

This document is structured as follows. In Section 2, we introduce the mathematical preliminaries of NCG, defining the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and illustrating how the choice of algebra dictates the geometry. In Section 3, we construct the discrete Dirac quantum walk in both free space and under a magnetic field. We then prove in Section 4 that this discrete evolution respects relativistic causality [8]. Finally, in Section 5, we evaluate the Connes distance for this system. We begin with the continuous setting, calculating the distance between Gaussian states first in a commutative algebra (recovering classical Euclidean distance) and then in a fully noncommutative algebra. We then switch to the discrete setting, estimating the distance under both commutative and noncommutative limits.

2 Mathematical Preliminaries

2.1 NCG and Connes distance

Classical geometry is built on the concept of a space formed by a set of points. Observables in this classical framework are continuous, complex functions defined on these points. The Gelfand-Naimark theorem shows that this point of view can be reversed: a space can be entirely reconstructed from the algebra of continuous functions defined on it. In other words, knowing all the functions on a space is equivalent to knowing the space itself. Or more precisely, every commutative C^*-algebra can be seen as the algebra of continuous functions on some topological space, and this correspondence is one to one and preserves structure. In this setting, the pointwise product of any two observables is commutative ($f \cdot g = g \cdot f$).

Alain Connes developed Noncommutative Geometry (NCG) by taking this algebraic translation a step further. If we replace the commutative algebra with a noncommutative algebra which reflects quantum mechanics where generally observables do not commute ($A \cdot B \neq B \cdot A$), the classical notion of sharply localized points breaks down due to the Heisenberg uncertainty principle. We lose the continuous manifold of points, but the algebraic structure remains, allowing us to still "do geometry" even without an underlying set of points.

Without classical points, the geometric space must be studied using "states". Mathematically, a state ϕ (a state functional and not a state/vector on a Hilbert space) is defined as a normalized, positive linear functional on the algebra. Instead of evaluating a classical function at a specific point x (giving $f(x)$), we evaluate an operator $a \in \mathcal{A}$ through its expectation value in a specific state, denoted $\phi(a)$. This matches the physical interpretation of quantum systems, where states describe distributions rather than exact positions and objects such as Gaussian wave packets can be viewed as the closest analogue to "locations" in a noncommutative space.

To introduce a notion of distance, Connes defined the concept of a spectral triple, denoted as $(\mathcal{A}, \mathcal{H}, D)$:

- $\mathcal{A}$: A C^*-algebra of bounded operators. The choice of this algebra (whether it commutes or not) defines the underlying topology and the "space" itself.
- $\mathcal{H}$: A Hilbert space over which the elements of $\mathcal{A}$ act as bounded linear operators.
- D: A selfadjoint Dirac operator with compact resolvent. It encodes the differential structure and acts as the fundamental "ruler" or metric of the space.

(In general, $\mathcal{A}$ can be any involutive algebra of operators such that the commutators $[D, a] = Da - aD$ are bounded for any $a \in \mathcal{A}$)

The classical distance between two points is generalized by the Connes distance between two states ϕ and ψ:

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

This expression can be understood as measuring how well the two states can be distinguished using observables with bounded variation. The condition $\|[D, a]\| \leq 1$ is analogue to a Lipschitz continuity bound. Because the Dirac operator behaves as a first order differential operator, its commutator with a captures how rapidly (the "gradient") the observable varies. Bounding this gradient restricts how "fast" the test operator can vary across the space, which ensures that the supremum remains finite.

2.2 Examples: Two point space and Qubit space

These two simple examples aim to illustrate how the change of the algebra affects the geometry. The spectral triple is :

- $\mathcal{H} = \mathbb{C}^2$.
- $D = m\sigma_x = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$.
- $\mathcal{A}$: We will use two different algebras.

2.2.1 Two point space

We restrict our algebra to purely diagonal matrices:

$$\mathcal{A} = \left\{ a = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} : h_1, h_2 \in \mathbb{C} \right\}$$

Because all diagonal matrices commute, $\mathcal{A}$ is a commutative algebra. We define $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The pure states on this algebra (state functionals) are:

$$\langle \uparrow |a| \uparrow \rangle = h_1$$
$$\langle \downarrow |a| \downarrow \rangle = h_2$$

Let us evaluate the commutator $[D, a]$:

$$\begin{aligned} [D, a] &= \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} - \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & mh_2 \\ mh_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & mh_1 \\ mh_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & m(h_2 - h_1) \\ m(h_1 - h_2) & 0 \end{pmatrix} \end{aligned}$$

Let $C = [D, a]$. Then:

$$C^\dagger C = |m|^2 |h_1 - h_2|^2 I$$

Therefore,

$$\|[D, a]\| = |m|\, |h_1 - h_2|$$

The constraint $\|[D, a]\| \leq 1$ becomes:

$$|h_1 - h_2| \leq \frac{1}{|m|}$$

Now compute the distance:

$$\begin{aligned} d_D(\uparrow, \downarrow) &= \sup_{h_1, h_2} \left\{ |h_1 - h_2| : |h_1 - h_2| \leq \frac{1}{|m|} \right\} \\ &= \frac{1}{|m|} \end{aligned}$$

Thus, the geometry consists of two discrete points separated by a distance $1/|m|$.

2.2.2 Qubit space

Now $\mathcal{A} = M_2(\mathbb{C})$ is a noncommutative algebra.
Any element can be expressed as:

$$a = h_0 I + \vec{h} \cdot \vec{\sigma} = h_0 I + h_x \sigma_x + h_y \sigma_y + h_z \sigma_z$$

where $h_0 \in \mathbb{R}$ and $\vec{h} \in \mathbb{R}^3$.
We define $\rho_{\vec{u}}$ and $\rho_{\vec{v}}$ the density matrices associated to $\vec{u}$ and $\vec{v}$. (if $\|\vec{u}\| = \|\vec{v}\| = 1$, then they are pure states).

$$\begin{aligned} \rho_{\vec{u}} &= \frac{1}{2}(I + \vec{u} \cdot \vec{\sigma}) \\ \rho_{\vec{v}} &= \frac{1}{2}(I + \vec{v} \cdot \vec{\sigma}) \end{aligned}$$

Therefore

$$\begin{aligned} \langle u|a|u\rangle &= \mathrm{Tr}(\rho_{\vec{u}} a) = h_0 + \vec{u} \cdot \vec{h} \\ \langle v|a|v\rangle &= \mathrm{Tr}(\rho_{\vec{v}} a) = h_0 + \vec{v} \cdot \vec{h} \end{aligned}$$

The difference is given by

$$\langle u|a|u\rangle - \langle v|a|v\rangle = (\vec{u} - \vec{v}) \cdot \vec{h}$$

The commutator

$$\begin{aligned}[D,a] &= [m\sigma_x, a] \\ &= m\big(h_y[\sigma_x,\sigma_y] + h_z[\sigma_x,\sigma_z]\big) \\ &= m\big(2ih_y\sigma_z - 2ih_z\sigma_y\big) \\ &= 2im(h_y\sigma_z - h_z\sigma_y)\end{aligned}$$

Then

$$C^\dagger C = 4m^2(h_y^2 + h_z^2)I$$

So

$$\|[D,a]\| = 2|m|\sqrt{h_y^2 + h_z^2}$$

Constraint

$$h_y^2 + h_z^2 \le \frac{1}{4|m|^2}$$

h_x is unconstrained.
Thus

$$d_D(\vec{u},\vec{v}) = \sup\left\{|(\vec{u}-\vec{v})\cdot\vec{h}| : h_y^2 + h_z^2 \le \frac{1}{4|m|^2}\right\}$$

If $u_x \neq v_x$, the distance is infinite.
If $u_x = v_x$, then

$$d_D(\vec{u},\vec{v}) = \frac{1}{2|m|}\sqrt{(u_y - v_y)^2 + (u_z - v_z)^2}$$

Switching to a noncommutative algebra changes the structure of the space. States that differ along the x axis ($u_x \neq v_x$) become infinitely distant. This means that the geometry decomposes into disjoint, inaccessible planes. The metric is no longer a simple measure of coordinate distance.

3 Dirac Quantum Walk

3.1 Discretization of a free Dirac Hamiltonian

The continuous Hamiltonian of a free Dirac particle in 2+1 dimensions (two spatial and one time dimension) is given by:

$$D_{free} = m\sigma_y + \sigma_x p_y + \sigma_z p_x$$

The time evolution of this system over a duration t is governed by the unitary operator $W = e^{-itD_{free}}$. To build a DTQW (discrete time quantum walk), we discretize the time into intervals of a very small time step ϵ, such that $t = N\epsilon$.

Substituting the momentum operators $p_x = -i\partial_x$ and $p_y = -i\partial_y$, the single step evolution operator W_ϵ becomes:

$$\begin{aligned} W_\epsilon &= e^{-i\epsilon(m\sigma_y + \sigma_x p_y + \sigma_z p_x)} \\ &= e^{-im\epsilon\sigma_y - \epsilon\sigma_x\partial_y - \epsilon\sigma_z\partial_x} \end{aligned}$$

Since the Pauli matrices do not commute, we apply a first order trotterization to separate the terms:

$$\begin{aligned} W_\epsilon &\approx e^{-im\epsilon\sigma_y}\ e^{-\epsilon\sigma_x\partial_y}\ e^{-\epsilon\sigma_z\partial_x} \\ &= M_\epsilon \cdot (HT_yH) \cdot T_x \end{aligned}$$

Here, $M_\epsilon = e^{-im\epsilon\sigma_y}$ is a coin rotation that depends on the mass, and H is the Hadamard operator used to change the basis:

$$HT_yH = e^{-\epsilon\sigma_x\partial_y} \iff T_y = e^{-\epsilon\sigma_z\partial_y}$$

The operators T_x and T_y are conditional shift operators on the joint Hilbert space of the position and the internal coin (spin) state. To map this continuous translation onto a discrete grid, we link the spatial lattice spacing to the time step such that $\Delta x = \Delta y = \epsilon$. Therefore, a physical translation by a distance of ϵ corresponds to an index shift of exactly 1. The action of T_y translates the particle along the y axis depending on its coin state:

$$\begin{aligned} T_y \left|x, y\right\rangle \otimes \left|0\right\rangle &= e^{-\epsilon\partial_y} \left|x, y\right\rangle \otimes \left|0\right\rangle \approx \left|x, y-1\right\rangle \otimes \left|0\right\rangle \\ T_y \left|x, y\right\rangle \otimes \left|1\right\rangle &= e^{+\epsilon\partial_y} \left|x, y\right\rangle \otimes \left|1\right\rangle \approx \left|x, y+1\right\rangle \otimes \left|1\right\rangle \end{aligned}$$

Similarly, the operator $T_x = e^{-\epsilon\sigma_z\partial_x}$ gives a conditional translation along the x axis:

$$\begin{aligned} T_x \left|x, y\right\rangle \otimes \left|0\right\rangle &= e^{-\epsilon\partial_x} \left|x, y\right\rangle \otimes \left|0\right\rangle \approx \left|x-1, y\right\rangle \otimes \left|0\right\rangle \\ T_x \left|x, y\right\rangle \otimes \left|1\right\rangle &= e^{+\epsilon\partial_x} \left|x, y\right\rangle \otimes \left|1\right\rangle \approx \left|x+1, y\right\rangle \otimes \left|1\right\rangle \end{aligned}$$

Notice that the shift operators T_x and T_y commute ($[T_x, T_y] = 0$). This comes from the fact that the underlying momentum operators p_x and p_y commute. Therefore, the order of the spatial translations in the x and y directions can be interchanged without modifying the dynamics of the quantum walk.

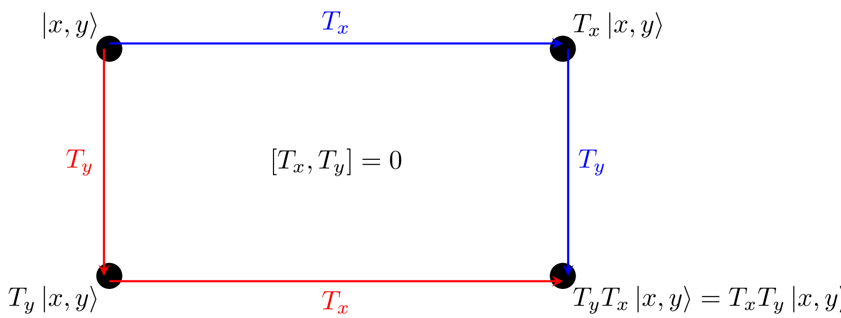


Figure 1: Diagram demonstrating the commutativity of spatial shifts.

3.2 Discretization of a Dirac Hamiltonian in a nonzero magnetic field

Let's consider now a Dirac particle still in 2+1 dimensions coupled to a magnetic field. The momentum operators are modified by a gauge vector potential $\vec{A} = (A_x(y), A_y(x))$. Therefore, the Hamiltonian becomes:

$$D = \Pi_x\sigma_z + \Pi_y\sigma_x + m\sigma_y$$

where

$$\begin{aligned}\Pi_x &= p_x - A_x(y)\\ \Pi_y &= p_y - A_y(x)\end{aligned}$$

In the limit where $A_x(y) = A_y(x) = 0$, we recover the Hamiltonian of a free particle.

Using the same logic as in the case of a free Hamiltonian, W_ϵ becomes:

$$\begin{aligned}W_\epsilon &= e^{-i\epsilon(\Pi_x\sigma_z+\Pi_y\sigma_x+m\sigma_y)}\\ &\approx e^{-im\epsilon\sigma_y}\, e^{-i\epsilon\Pi_y\sigma_x}\, e^{-i\epsilon\Pi_x\sigma_z}\\ &= M_\epsilon\cdot(HT_yH)\cdot T_x\end{aligned}$$

The mass operator M_ϵ remains identical to the free particle case, since the magnetic field couples only to the momentum and does not affect the mass term. However, the shift operators are now modified by the gauge vector potential:

$$\begin{aligned}T_x &= e^{-i\epsilon\Pi_x\sigma_z} = e^{-\epsilon\partial_x+i\epsilon A_x(y)\sigma_z}\\ T_y &= e^{-i\epsilon\Pi_y\sigma_z} = e^{-\epsilon\partial_y+i\epsilon A_y(x)\sigma_z}\end{aligned}$$

Their action is given by:

$$T_y |x,y\rangle \otimes |0\rangle = e^{-\epsilon\partial_y + i\epsilon A_y(x)} |x,y\rangle \otimes |0\rangle \approx e^{i\epsilon A_y(x)} |x,y-1\rangle \otimes |0\rangle$$
$$T_y |x,y\rangle \otimes |1\rangle = e^{+\epsilon\partial_y - i\epsilon A_y(x)} |x,y\rangle \otimes |1\rangle \approx e^{-i\epsilon A_y(x)} |x,y+1\rangle \otimes |1\rangle$$

$$T_x |x,y\rangle \otimes |0\rangle = e^{-\epsilon\partial_x + i\epsilon A_x(y)} |x,y\rangle \otimes |0\rangle \approx e^{i\epsilon A_x(y)} |x-1,y\rangle \otimes |0\rangle$$
$$T_x |x,y\rangle \otimes |1\rangle = e^{+\epsilon\partial_x - i\epsilon A_x(y)} |x,y\rangle \otimes |1\rangle \approx e^{-i\epsilon A_x(y)} |x+1,y\rangle \otimes |1\rangle$$

Unlike the free particle case, the shift operators T_x and T_y do not commute (the gauge vector potential A depend purely on the position which do not commute with the momentum). When they are applied in sequence, the coin states accumulate position dependent gauge phases. For instance, comparing the action of T_yT_x with T_xT_y on a basis state $|x,y\rangle \otimes |0\rangle$ gives:

$$T_yT_x |x,y\rangle \otimes |0\rangle \approx e^{i\epsilon[A_x(y)+A_y(x-1)]} |x-1,y-1\rangle \otimes |0\rangle$$
$$T_xT_y |x,y\rangle \otimes |0\rangle \approx e^{i\epsilon[A_y(x)+A_x(y-1)]} |x-1,y-1\rangle \otimes |0\rangle$$

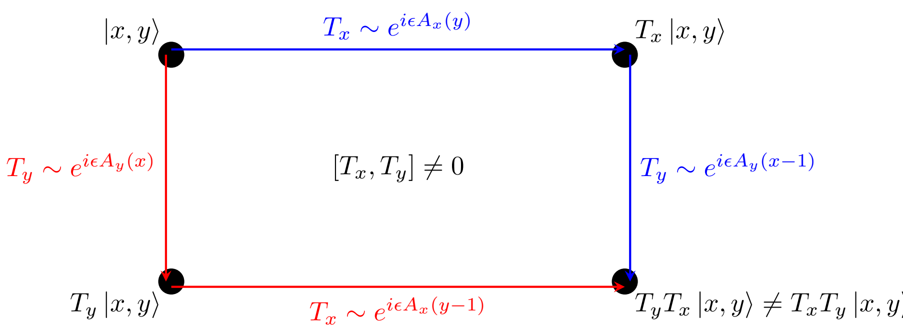


Figure 2: Diagram demonstrating the noncommutativity of spatial shifts

4 Relativistic Causality

After establishing the unitary evolution of the discrete Dirac quantum walk, a key requirement is the preservation of relativistic causality. In the continuous regime, the Dirac equation strictly enforces that information propagates within a causal light cone. However, when switching to a discrete lattice, or when dealing with standard non relativistic quantum models (such as the free

Schrodinger equation), localized wavepackets often spread instantly, violating strict causality. To ensure that the DTQW remains relativistically consistent, even in the presence of a magnetic field, we prove that the dynamics satisfy a Lieb–Robinson type bound. In this section, we will demonstrate that the spatial support of any localized observable expands by at most one lattice unit per time step, therefore confirming the existence of a discrete causal light cone.

4.1 Definitions

The system is defined on the Hilbert space

$$\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathcal{H}_c$$

where $\ell^2(\mathbb{Z}^2)$ is spanned by the position basis $|r\rangle$ for $r \in \mathbb{Z}^2$, and $\mathcal{H}_c \cong \mathbb{C}^2$ is the coin space spanned by $\{|0\rangle, |1\rangle\}$.

Let $\mathcal{A}$ be the C^*-algebra of bounded operators on $\mathcal{H}$.
An operator $A \in \mathcal{A}$ is localized in a spatial region $\mathcal{R} \subset \mathbb{Z}^2$ if it can be written as

$$A = A_{\mathcal{R}} \otimes \mathbb{I}_{\mathbb{Z}^2 \setminus \mathcal{R}}$$

where $A_{\mathcal{R}}$ acts non trivially (different than the identity) only on the sites in $\mathcal{R}$.

The support of an operator A, supp(A), is the minimal set $\mathcal{R} \subset \mathbb{Z}^2$ such that A acts as the identity on its complement.

4.2 Walk Decomposition

A single step of our evolution operator (as mentionned above) is:

$$W_\epsilon = M_\epsilon \cdot (HT_yH) \cdot T_x$$

We decompose this into local and shift operators. Let

$$P_0 = |0\rangle\langle 0|, \qquad P_1 = |1\rangle\langle 1|$$

the projectors in the coin space (corresponding to σ_z eigenstates). The shifts T_x and T_y can be decomposed into gauge phases D and conditional shifts S.

Local Coins (C):

$$C_1 = H, \qquad C_2 = M_\epsilon H$$

These act purely on $\mathcal{H}_c$ at each site.

Gauge Phases (D):

$$D_j = \sum_r |r\rangle\langle r| \otimes e^{i\epsilon A_j(r)\sigma_z}$$

These are diagonal operators which apply local phases.

Shifts (S):

$$S_x = \sum_r \left(|r - e_x\rangle \langle r| \otimes P_0 + |r + e_x\rangle \langle r| \otimes P_1 \right)$$

$$S_y = \sum_r \left(|r - e_y\rangle \langle r| \otimes P_0 + |r + e_y\rangle \langle r| \otimes P_1 \right)$$

Since we are in 2D, $r = \begin{pmatrix} x \\ y \end{pmatrix}$, $e_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Thus, we rewrite the step as

$$W_\epsilon = C_2 \cdot D_y \cdot S_y \cdot C_1 \cdot D_x \cdot S_x$$

4.3 Walk Causality

Recall that time was discretized such that $t = N\epsilon$. For a local observable A_0 at the origin (such that $\text{supp}(A_0) = \{0\}$), we want to prove that the evolved operator

$$A(N) = ({W_\epsilon}^\dagger)^N A_0 {W_\epsilon}^N$$

has its support strictly bounded by the causal light cone:

$$\text{supp}(A(N)) \subseteq \left\{ r \in \mathbb{Z}^2 \;\middle|\; \|r\|_\infty \leq N \right\}$$

4.3.1 Lemmas

Lemma 1: Coins Let

$$C = \sum_r |r\rangle \langle r| \otimes U_r$$

a unitary operator acting locally on the coin space. If $\text{supp}(A) \subseteq \mathcal{R}$, then

$$\text{supp}(C^\dagger A C) \subseteq \mathcal{R}$$

Proof: Because C is block diagonal in the position basis, it commutes with the spatial projectors $|r\rangle \langle r|$. For any A localized in $\mathcal{R}$, $C^\dagger A C$ only mixes the internal coin degrees of freedom within $\mathcal{R}$. It acts as the identity on any $r' \notin \mathcal{R}$. Thus, the spatial support is invariant.

Lemma 2: Gauge Phases Let

$$D = \sum_r |r\rangle \langle r| \otimes e^{i\Phi(r)\sigma_z}$$

If $\text{supp}(A) \subseteq \mathcal{R}$, then

$$\text{supp}(D^\dagger A D) \subseteq \mathcal{R}$$

Proof: D is diagonal in the position basis. Let

$$A = \sum_{r,r' \in \mathcal{R}} |r\rangle \langle r'| \otimes O_{r,r'}$$

Conjugating by D gives

$$D^\dagger A D = \sum_{r,r' \in \mathcal{R}} |r\rangle \langle r'| \otimes \left(e^{-i\Phi(r)\sigma_z} \, O_{r,r'} \, e^{i\Phi(r')\sigma_z} \right)$$

This operation only modifies the coin operator amplitudes using local phases, it does not generate terms outside of $\mathcal{R}$.

Lemma 3: Shifts Let S_x and S_y be the conditional shift along the x and y axes. If $\text{supp}(A) \subseteq \mathcal{R}$, then

$$\text{supp}(S_x^\dagger A S_x) \subseteq (\mathcal{R} + e_x) \cup (\mathcal{R} - e_x)$$

$$\text{supp}(S_y^\dagger A S_y) \subseteq (\mathcal{R} + e_y) \cup (\mathcal{R} - e_y)$$

Proof: Any observable supported in $\mathcal{R}$ can be written as a linear combination of operators

$$A = |r\rangle \langle r'| \otimes O_c, \qquad r, r' \in \mathcal{R}$$

Conjugating by S_x gives

$$S_x^\dagger A S_x = \sum_{i,j \in \{0,1\}} |r + (-1)^i e_x\rangle \langle r' + (-1)^j e_x| \otimes (P_i O_c P_j)$$

The non zero terms exist only for positions shifted by $\pm e_x$. The same logic applies to S_y expanding the support by $\pm e_y$.

4.3.2 Theorem

For the quantum walk W_ϵ, if $\text{supp}(A) \subseteq \mathcal{R}$, then after one time step

$$\text{supp}({W_\epsilon}^\dagger A W_\epsilon) \subseteq \{ r + \delta \mid r \in \mathcal{R}, \ \|\delta\|_\infty \leq 1 \}$$

Proof: Let A_0 be supported on $\mathcal{R}$. One step is

$$A_1 = S_x^\dagger D_x^\dagger C_1^\dagger S_y^\dagger D_y^\dagger C_2^\dagger A_0 C_2 D_y S_y C_1 D_x S_x$$

We apply sequentially the lemmas proved above:

- C_2 and D_y: the support remains $\mathcal{R}$.
- S_y: the support expands by at most $\pm e_y$. Let this new region be $\mathcal{R}_y$.
- C_1 and D_x: the support remains $\mathcal{R}_y$.
- S_x: the support expands by at most $\pm e_x$.

Thus, after one step, the maximum distance any term has spread from $\mathcal{R}$ is one unit in y and one unit in x. Therefore,

$$\text{supp}(A_1) \subseteq \{r + \delta_x e_x + \delta_y e_y \mid r \in \mathcal{R},\ \delta_x, \delta_y \in \{-1, 1\}\}$$

Induction: By induction, if A_0 is supported at the origin at $t = 0$, then

$$\text{supp}(A(N)) \subseteq \left\{r \in \mathbb{Z}^2 \mid \|r\|_\infty \leq N\right\}$$

Therefore, for any local observable $B_{r'}$ supported at r', we have

$$[A(N), B_{r'}] = 0 \qquad \forall\ \|r'\|_\infty > N$$

Thus, the walk is causal.

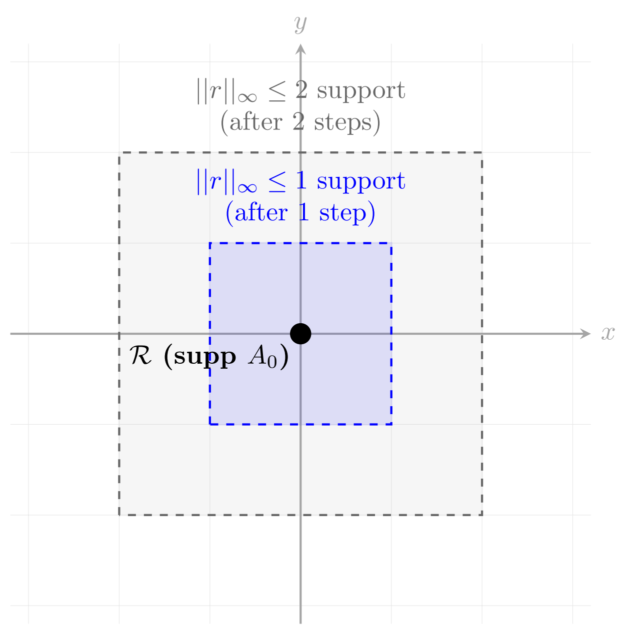


Figure 3: The expansion of the support

5 Connes Distance

5.1 The Continuous Framework

5.1.1 Commutative Algebra

5.1.1.1 Spatial distance: We define the spectral triple $(\mathcal{A}, \mathcal{H}, D)$:

- $\mathcal{A}$: the commutative C^*-algebra of essentially bounded, complex valued functions on space (i.e. $L^\infty(\mathbb{R}^2)$).

- $\mathcal{H}$: the Hilbert space $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$
- D: the Dirac operator defined by $D = \Pi_x \sigma_z + \Pi_y \sigma_x + m\sigma_y$ with $\Pi_j = -i\partial_j - A_j = p_j - A_j$.

The distance in NCG is given by:

$$d_D(\phi, \psi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$

with ϕ and ψ state functionals (states in the algebraic sense, i.e. expected value).

Computing the commutator:

$$[D, a] = [\sigma_z \Pi_x + \sigma_x \Pi_y + m\sigma_y, a] = [\sigma_z \Pi_x, a] + [\sigma_x \Pi_y, a] + [m\sigma_y, a]$$

We will use test operators that are purely scalar functions of position, taking the form $a = g(r) \otimes I_2$, so it commutes with the gauge potentials A_j (functions of position) and it commutes with the mass term (identity on the spin space):

$$[m\sigma_y, a] = 0$$

In order to calculate the two other commutators, we study how they act on a function f:

$$\begin{aligned}
[\sigma_x \Pi_y, a]f &= [-i\sigma_x \partial_y, a]f \\
&= -i\sigma_x(\partial_y af) + ia\sigma_x(\partial_y f) \\
&= -ia\sigma_x(\partial_y f) - if\sigma_x(\partial_y a) + ia\sigma_x(\partial_y f) \\
&= -if\sigma_x(\partial_y a)
\end{aligned}$$

which means $[\sigma_x \Pi_y, a] = -i\sigma_x(\partial_y a)$ and more generally

$$[\Pi_j, a] = [-i\partial_j, a] = -i\,(\partial_j a)$$

Thus

$$\begin{aligned}
[D, a] &= -i\sigma_z(\partial_x a) - i\sigma_x(\partial_y a) \\
&= -i \sum_{\mu \in 0,1} \gamma^\mu (\partial_\mu a) = -i\gamma \cdot \nabla a
\end{aligned}$$

with $\gamma^1 = \sigma_z$ and $\gamma^2 = \sigma_x$.

The condition over the norm of the commutator becomes

$$\|[D, a]\| \leq 1 \iff \| - i\gamma \cdot \nabla a\| = \sup |\nabla a| \leq 1$$

Let $|\zeta_1\rangle$ and $|\zeta_2\rangle$ be two gaussian states (in the usual sense, i.e, states on a Hilbert space) centered respectively at r_1 and r_2 with the same width σ. To compute the expectation value $\phi(a) = \langle\zeta_1|a|\zeta_1\rangle$, we use $I = \int d^2r\,|r\rangle\,\langle r|$:

$$\begin{aligned}
\phi(a) = \langle\zeta_1|\,a\,|\zeta_1\rangle &= \int d^2r\,\langle\zeta_1|\,a\,|r\rangle\,\langle r|\zeta_1\rangle \\
&= \int d^2r\,a(r)\,\langle\zeta_1|r\rangle\,\langle r|\zeta_1\rangle \\
&= \int a(r)\,|\zeta_1(r)|^2\,d^2r \\
\psi(a) = \langle\zeta_2|\,a\,|\zeta_2\rangle &= \int a(r)\,|\zeta_2(r)|^2\,d^2r
\end{aligned}$$

Since $|\zeta(r)|^2$ is a gaussian probability distribution, the Connes distance becomes

$$d_D(\zeta_1,\zeta_2) = \sup_{a:\,|\nabla a|\leq 1}\left|\int a(r)\,\rho_1(r)\,d^2r - \int a(r)\,\rho_2(r)\,d^2r\right|$$

where $\rho_i = |\zeta_i|^2$.
We introduce ρ_0 such that $\rho_i(r) = \rho_0(r - r_i)$. Changing variables to $x = r - r_i$ in each integral gives:

$$d_D(\zeta_1,\zeta_2) = \sup_{a:|\nabla a|\leq 1}\left|\int \Big[a(x+r_1) - a(x+r_2)\Big]\rho_0(x)d^2x\right|$$

The condition $|\nabla a| \leq 1$ implies $|a(x+r_1) - a(x+r_2)| \leq \|r_1 - r_2\|$. Therefore

$$d_D(\zeta_1,\zeta_2) \leq \int \|r_1 - r_2\|\rho_0(x)d^2x = \|r_1 - r_2\|$$

On the other hand, we choose $a(r) = \frac{r_1 - r_2}{\|r_1 - r_2\|}\cdot r$ which satisfies $|\nabla a| = 1$, and since d_D is the sup:

$$\begin{aligned}
d_D(\zeta_1,\zeta_2) &\geq \int a(r)\rho_1(r)d^2r - \int a(r)\rho_2(r)d^2r \\
&\geq a(r_1) - a(r_2) = \frac{r_1 - r_2}{\|r_1 - r_2\|}\cdot(r_1 - r_2) = \|r_1 - r_2\|
\end{aligned}$$

We used above the fact that

$$\begin{aligned}
\int a(r)\rho_1(r)d^2r - \int a(r)\rho_2(r)d^2r &= \int a(r_1)\rho_0(x)d^2x - \int a(r_2)\rho_0(x)d^2x \\
&= a(r_1)\int \rho_0(x)d^2x - a(r_2)\int \rho_0(x)d^2x \\
&= a(r_1) - a(r_2)
\end{aligned}$$

Thus, combining the lower and upper bound, we obtain the classical euclidean distance:

$$d_D(\zeta_1,\zeta_2) = \|r_1 - r_2\|$$

Since ζ_1 and ζ_2 are both gaussian states and have the same width, the only difference between them is the spatial distance between their centers. This is why the Connes distance is the Euclidean distance. If the two states had different widths, we would expect the formula to give additional terms describing the quantum "cost" of the compression or expansion.

5.1.1.2 Spin distance: Another simple example worth mentioning is the pure spin distance.

Consider two states at the same position (so spatial distance is 0), they share the same spatial wavepacket $|\alpha\rangle$, but with opposite spins: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (spin up) and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (spin down).

The full states in $\mathcal{H}$ are:

$$|\psi_\uparrow\rangle = |\alpha\rangle \otimes |\uparrow\rangle$$
$$|\psi_\downarrow\rangle = |\alpha\rangle \otimes |\downarrow\rangle$$

To distinguish them, we need a test operators that act purely on the internal coin space and has different values on the diagonal, taking the form

$$a = I \otimes \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} = I \otimes h\sigma_z$$

where h is a constant number. Notice that the algebra is commutative.

Because $\langle\alpha| I |\alpha\rangle = 1$, the difference in expectation values is simply

$$\langle\psi_\uparrow| a |\psi_\uparrow\rangle - \langle\psi_\downarrow| a |\psi_\downarrow\rangle = h - (-h) = 2h$$

The commutator becomes

$$\begin{aligned}
[D, a] &= [\sigma_z \Pi_x + \sigma_x \Pi_y + m\sigma_y, h\sigma_z] \\
&= h\Big([\sigma_z, \sigma_z]\Pi_x + [\sigma_x, \sigma_z]\Pi_y + m[\sigma_y, \sigma_z]\Big) \\
&= h\Big(0 - 2i\sigma_y \Pi_y + 2im\sigma_x\Big) \\
&= 2ih(m\sigma_x - \Pi_y \sigma_y)
\end{aligned}$$

Let $C = [D, a]$. We need to calculate $(C^\dagger C)^{\frac{1}{2}}$ to get norm of the commutator

$$\begin{aligned}
C^\dagger C &= 4|h|^2 (m\sigma_x - \Pi_y \sigma_y)(m\sigma_x - \Pi_y \sigma_y) \\
&= 4|h|^2 (m^2 \sigma_x^2 + \Pi_y^2 \sigma_y^2 - m\Pi_y \sigma_x \sigma_y - \Pi_y m \sigma_y \sigma_x) \\
&= 4|h|^2 \Big((m^2 + \Pi_y^2) I - m\Pi_y (\sigma_x \sigma_y + \sigma_y \sigma_x)\Big) \\
&= 4|h|^2 (m^2 + \Pi_y^2) I
\end{aligned}$$

Assuming we are in a subspace where the momentum operator Π_y has a bounded eigenvalue π_y, the operator norm is:

$$\|[D, a]\| = 2|h|\sqrt{m^2 + \pi_y^2}$$

Using the condition of Connes distance $\|[D, a]\| \leq 1$

$$2|h|\sqrt{m^2 + \pi_y^2} \leq 1 \implies |h| \leq \frac{1}{2\sqrt{m^2 + \pi_y^2}}$$

Therefore, using the difference in expectation values, we get

$$d_D(\psi_\uparrow, \psi_\downarrow) = \frac{1}{\sqrt{m^2 + \pi_y^2}}$$

Normally, one would expect the distance to be zero since both $|\psi_\uparrow\rangle$ and $|\psi_\downarrow\rangle$ share the exact same position. By using an algebra that includes the diagonal spin matrix, we have implicitly expanded our geometry from a simple 2D continuous plane into a space with two sides: $\mathbb{R}^2 \times \{\uparrow, \downarrow\}$. The two spin states live at the same spatial coordinates, but on different sides of this space. The Connes distance shows that the geometric separation between these two sides is not arbitrary, but is defined by the physical parameters of the Dirac operator.

5.1.2 Noncommutative Algebra

We've restricted, in the commutative case above, the test operator to purely scalar functions of position, taking the form $a = g(r) \otimes I_2$. This implies the search of a supremum in the algebra of bounded position functions, which is a commutative algebra, and much smaller than the full algebra of bounded operators.

In order to study the full algebra of bounded operator $\mathcal{A}$, we cannot use polynomials of position and momentum as a test operator because the momentum is unbounded, and therefore $a \notin \mathcal{A}$ anymore.

We will use

$$a = e^{i(k \cdot r + q \cdot p)} \otimes I_2 = e^{ik \cdot r} e^{iq \cdot p} e^{i(k \cdot q)/2} \otimes I_2$$

where r is the position, p is the momentum, and $k = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$ and $q = \begin{pmatrix} q_x \\ q_y \end{pmatrix}$ are real vectors.

Evaluating the commutator

Recall that our Dirac operator is: $D = \Pi_x \sigma_z + \Pi_y \sigma_x + m\sigma_y$ with $\Pi_j = -i\partial_j - A_j = p_j - A_j$.

Since a is the identity on the spin space, the mass term commutes:

$$[m\sigma_y, a] = 0$$

We need to compute $[\Pi_j, a] = [p_j, a] - [A_j(r), a]$. First, starting by:

$$\begin{aligned}
[p_j, e^{ik\cdot r}] &= [p_j, e^{ik\cdot r}] \\
&= -i\partial_j e^{i(k_x x + k_y y)} \\
&= -i(ik_j)e^{ik\cdot r} \\
&= k_j e^{ik\cdot r}
\end{aligned}$$

Using $[X, YZ] = [X, Y]Z + Y[X, Z]$:

$$\begin{aligned}
[p_j, a] &= [p_j, e^{ik\cdot r}e^{iq\cdot p}e^{i(k\cdot q)/2}] \\
&= [p_j, e^{ik\cdot r}]e^{iq\cdot p}e^{i(k\cdot q)/2} + e^{ik\cdot r}[p_j, e^{iq\cdot p}]e^{i(k\cdot q)/2} \\
&= (k_j e^{ik\cdot r})e^{iq\cdot p}e^{i(k\cdot q)/2} + 0 \\
&= k_j a
\end{aligned}$$

We evaluate the second commutator that depends on the gauge terms $[A_j(r), a]$. Since $iq \cdot p = q \cdot \nabla$, expanding $e^{q\cdot\nabla}$ as a taylor series acting on a function $f(r)$:

$$e^{q\cdot\nabla} f(r) = \sum_{n=0}^{\infty} \frac{(q\cdot\nabla)^n}{n!} f(r)$$

By definition, this is the expansion of $f(r + q)$. Therefore:

$$e^{iq\cdot p} f(r) = f(r + q)$$

If we apply $e^{iq\cdot p}A_j(r)$ to a wavepacket $\psi(r)$, the translation operator shifts everything to its right:

$$e^{iq\cdot p}\big(A_j(r)\psi(r)\big) = A_j(r+q)\psi(r+q) = A_j(r+q)\big(e^{iq\cdot p}\psi(r)\big)$$

$$\iff e^{iq\cdot p}A_j(r) = A_j(r+q)e^{iq\cdot p}$$

Since $A_j(r)$ commutes with the position exponential $e^{ik\cdot r}$

$$\begin{aligned}
[A_j(r), a] &= A_j(r)a - aA_j(r) \\
&= A_j(r)a - e^{ik\cdot r}e^{iq\cdot p}e^{i(k\cdot q)/2}A_j(r) \\
&= A_j(r)a - e^{ik\cdot r}A_j(r+q)e^{iq\cdot p}e^{i(k\cdot q)/2} \\
&= A_j(r)a - A_j(r+q)a \\
&= \big(A_j(r) - A_j(r+q)\big)a
\end{aligned}$$

We conclude that

$$[\Pi_j, a] = k_j a - \big(A_j(r) - A_j(r+q)\big)a = V_j(r)a$$

with $V_j(r) = k_j - A_j(r) + A_j(r+q)$

Therefore

$$[D,a] = \big(V_x(r)\sigma_z + V_y(r)\sigma_x\big)a$$

If the magnetic field is uniform (which we will assume later), then the commutator is bounded. But for exponential or quadratic magnetic fields (non linear), it is unbounded unless $A_j(r+q) - A_j(r)$ is bounded in r which is a strong assumption.

Computing the norm $\|[D,a]\|$:

$$\begin{aligned}
[D,a]^\dagger[D,a] &= a^\dagger\Big(V_x(r)\sigma_z + V_y(r)\sigma_x\Big)^2 a \\
&= a^\dagger\Big(V_x(r)^2 + V_y(r)^2\Big)\, I_2\ a \\
&= a^\dagger\Big(V_x(r)^2 + V_y(r)^2\Big)\ a \\
&= e^{-i(k\cdot q)/2}e^{-iq\cdot p}e^{-ik\cdot r}\ \Big(V_x(r)^2 + V_y(r)^2\Big)\ e^{ik\cdot r}e^{iq\cdot p}e^{i(k\cdot q)/2} \\
&= e^{-iq\cdot p}\Big(V_x(r)^2 + V_y(r)^2\Big)e^{iq\cdot p} \\
&= V_x(r-q)^2 + V_y(r-q)^2
\end{aligned}$$

Therefore,

$$\|[D,a]\| = \sup_{r\in\mathbb{R}^2}\sqrt{V_x(r)^2 + V_y(r)^2} = \sup_{r\in\mathbb{R}^2}\sqrt{\big(k_x - A_x(r) + A_x(r+q)\big)^2 + (k_y - A_y(r) + A_y(r+q))^2}$$

5.1.2.1 Free Dirac (no magnetic field)

Taking $A_x = A_y = 0$, the condition becomes:

$$\|[D,a]\| = \|k\| \le 1$$

Choosing $a = \frac{1}{\|k\|}e^{ik\cdot r}e^{iq\cdot p}e^{i(k\cdot q)/2}\otimes I_2$, we saturate the condition above: $\|[D,a]\| = 1$.

Changing the variables $x = r - r_i$, the state functionals, as defined above for the gaussian states, become:

$$\begin{aligned}
\phi(a) = \langle\zeta_1|\,a\,|\zeta_1\rangle &= \frac{1}{\|k\|}e^{i(k\cdot q)/2}\int \zeta_1^*(r)e^{ik\cdot r}\zeta_1(r+q)d^2r \\
&= e^{ik\cdot r_1}\left[e^{i(k\cdot q)/2}\int \zeta_0^*(x)e^{ik\cdot x}\zeta_0(x+q)d^2x\right] \\
\psi(a) = \langle\zeta_2|\,a\,|\zeta_2\rangle &= e^{ik\cdot r_2}\left[e^{i(k\cdot q)/2}\int \zeta_0^*(x)e^{ik\cdot x}\zeta_0(x+q)d^2x\right]
\end{aligned}$$

Therefore, the Cnones distance is expressed as:

$$d_D(\zeta_1,\zeta_2) = \sup_{k,q}\frac{1}{\|k\|}|e^{ik\cdot r_1} - e^{ik\cdot r_2}|\ \underbrace{|e^{i(k\cdot q)/2}\int \zeta_0^*(x)e^{ik\cdot x}\zeta_0(x+q)d^2x|}_{X(k,q)}$$

We want to find the maximum of $X(k,q)$ over q. The overlap $\langle\zeta_0(x)|\zeta_0(x+q)\rangle$ is maximized when $q=0$, and consequently, the integral becomes the fourier transform of the probability density. Thus, assuming ζ_0 is a normalized gaussian with width σ, $\zeta_0(x) = \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{|x|^2}{2\sigma^2}}$

$$\begin{aligned} d_D(\zeta_1,\zeta_2) &= \sup_k \frac{1}{\|k\|} |e^{ik\cdot r_1} - e^{ik\cdot r_2}| \, |\int e^{ik\cdot x} \|\zeta_0(x)\|^2 d^2x| \\ &= \sup_k \frac{1}{\|k\|} |e^{ik\cdot r_1} - e^{ik\cdot r_2}| \, |e^{-\frac{\sigma^2\|k\|^2}{4}}| \end{aligned}$$

$X(k,0)$ is maximized when $k=0$. Therefore, taking the limit when $k\to 0$, or decomposing it as a constant times a unit vector $k=\alpha\vec{n}$ when $\alpha\to 0$

$$\lim_{\alpha\to 0} \frac{1}{\alpha} |e^{i\alpha\vec{n}\cdot r_1} - e^{i\alpha\vec{n}\cdot r_2}| = |i\alpha\vec{n}\cdot(r_1-r_2)|$$

To maximize this dot product, we choose $\vec{n} = r_1 - r_2$ which gives $\|r_1 - r_2\|$

5.1.2.2 Non zero uniform magnetic field We introduce a uniform magnetic field B. We choose the Landau gauge defined by:

$$A_x(r) = By, \qquad A_y(r) = 0$$

Substituting in $V_j(r)$:

$$\begin{aligned} V_x(r) &= k_x - By + B(y+q_y) = k_x + Bq_y \\ V_y(r) &= k_y - 0 + 0 = k_y \end{aligned}$$

The spatial coordinate y cancels out. The norm of the Dirac commutator becomes independent of position:

$$\|[D,a]\| = \sup_{r\in\mathbb{R}^2} \sqrt{(k_x+Bq_y)^2 + k_y^2} = \sqrt{(k_x+Bq_y)^2 + k_y^2}$$

The spatial translation parameter along the x axis, q_x, is absent from the constraint which represents an unconstrained degree of freedom.
The momentum shift k_x and the spatial translation q_y are coupled by the magnetic field B.

Let $N(k,q) = \sqrt{(k_x+Bq_y)^2 + k_y^2}$. To saturate the bound in the constraint $\|[D,a]\| \le 1$, we choose our test operator $a = \frac{1}{N(k,q)} e^{i(k\cdot r + q\cdot p)} \otimes I_2$.
Using the same logic and derivation for the state functionals evaluated on the gaussian wavepackets ζ_1 and ζ_2 centered at r_1 and r_2, the Connes distance becomes:

$$d_D(\zeta_1,\zeta_2) = \sup_{k,q} \frac{1}{N(k,q)} |e^{ik\cdot r_1} - e^{ik\cdot r_2}| \, |X(k,q)|$$

where $X(k,q) = e^{i(k\cdot q)/2}\int \zeta_0^*(x)e^{ik\cdot x}\zeta_0(x+q)d^2x$.
Unlike the previous case where $B=0$, the supremum is no longer maximized by $q=0$. Because the denominator $N(k,q)$ depends on the entangled term $(k_x+Bq_y)^2$, the algebra permits test operators with large spatial translations q_y if they are compensated by a momentum change $k_x \approx -Bq_y$. Assuming this condition, the denominator is bounded by $|k_y|$, which prevents the fraction from decaying.

Therefore, the distance is no longer the euclidean one $\|r_1 - r_2\|$.

5.2 The Discrete Framework

5.2.1 Commutative Algebra

The discrete quantum walk is given by:

$$W_\epsilon \approx e^{-i\epsilon D} \approx I - i\epsilon D$$

Therefore

$$D = \frac{i}{\epsilon}(W_\epsilon - I)$$

$$\begin{aligned}[D,a] &= \frac{i}{\epsilon}[W_\epsilon - I, a] \\ &= \frac{i}{\epsilon}[W_\epsilon, a] \\ &= \frac{i}{\epsilon}(W_\epsilon a - aW_\epsilon) \\ &= \frac{i}{\epsilon}W_\epsilon(a - W_\epsilon^\dagger a W_\epsilon)\end{aligned}$$

We take the commutator norm and because W_ϵ is unitary

$$\begin{aligned}\|[D,a]\| &= \frac{1}{\epsilon}\|W_\epsilon(a - W_\epsilon^\dagger a W_\epsilon)\| \\ &= \frac{1}{\epsilon}\|a - W_\epsilon^\dagger a W_\epsilon\| \\ &= \frac{1}{\epsilon}\|[W_\epsilon, a]\|\end{aligned}$$

Thus, the Connes distance condition becomes in the discrete case:

$$\|[D,a]\| \le 1 \iff \frac{1}{\epsilon}\|[W_\epsilon,a]\| \le 1 \iff \|[W_\epsilon,a]\| \le \epsilon$$

Definitions in the discrete case

We restrict our algebra $\mathcal{A}$ to the commutative C^*-algebra of bounded purely spatial functions. An operator $a \in \mathcal{A}$ is diagonal in the position basis and acts as the identity on the coin space:

$$a = \sum_{r\in\mathbb{Z}^2} a_r \left|r\right\rangle\left\langle r\right| \otimes I_2$$

where $a_r \in \mathbb{R}$.

Let $\zeta_1 = |r_1\rangle \otimes |c\rangle$ and $\zeta_2 = |r_2\rangle \otimes |c\rangle$ be two states on the lattice. Their functionals are simply:

$$\phi(a) = \langle r_1, c| \, a \, |r_1, c\rangle = a_{r_1}$$
$$\psi(a) = \langle r_2, c| \, a \, |r_2, c\rangle = a_{r_2}$$

Using the discrete norm condition, the Connes distance between two lattice sites is defined as:

$$d_{W_\epsilon}(r_1, r_2) = \sup_{a \in \mathcal{A}} \Big\{ |a_{r_1} - a_{r_2}| : \|[W_\epsilon, a]\| \leq \epsilon \Big\}$$

The Commutator

We study the action of W_ϵ on a basis state $|r\rangle \otimes |c\rangle$, where $r = (x, y)$.

Recall the decomposition from the Causality section: $W_\epsilon = C_2 D_y S_y C_1 D_x S_x$. Based on the causality proof, the shift operator S_x shifts the spatial support by $\pm e_x$, and S_y shifts it by $\pm e_y$. The coin operators C_j and gauge phases D_j only modify the internal coin amplitudes and apply local phases, but do not change the position.

Therefore, a single step of the walk creates a superposition over the four diagonal neighbors:

$$W_\epsilon\big(|r\rangle \otimes |c\rangle \big) = \sum_{s_x \in \{-1,1\}} \sum_{s_y \in \{-1,1\}} |r + s_x e_x + s_y e_y\rangle \otimes U_{s_x, s_y}(r) \, |c\rangle$$

where $U_{s_x,s_y}(r)$ are 2×2 matrices/unitaries. Let $\delta = (s_x, s_y)$ represent an allowed jump vector, and $\mathcal{N}$ be the set of the four diagonal jumps $\mathcal{N} = \{(\pm 1, \pm 1)\}$. We rewrite this as:

$$W_\epsilon\big(|r\rangle \otimes |c\rangle \big) = \sum_{\delta \in \mathcal{N}} |r + \delta\rangle \otimes U_\delta(r) \, |c\rangle$$

Now, we apply the test operator $a = \sum_{r'} a_{r'} \, |r'\rangle \langle r'| \otimes I_c$. Since a is diagonal in the spatial basis, it simply multiplies the state by the scalar $a_{r'}$ corresponding to its position.

$$W_\epsilon a\big(|r\rangle \otimes |c\rangle \big) = a_r W_\epsilon\big(|r\rangle \otimes |c\rangle \big) = \sum_{\delta \in \mathcal{N}} a_r \, |r + \delta\rangle \otimes U_\delta(r) \, |c\rangle$$

$$a W_\epsilon\big(|r\rangle \otimes |c\rangle \big) = a \left(\sum_{\delta \in \mathcal{N}} |r + \delta\rangle \otimes U_\delta(r) \, |c\rangle \right) = \sum_{\delta \in \mathcal{N}} a_{r+\delta} \, |r + \delta\rangle \otimes U_\delta(r) \, |c\rangle$$

Subtracting these two terms gives the action of the commutator:

$$[W_\epsilon, a]\big(|r\rangle \otimes |c\rangle \big) = \sum_{\delta \in \mathcal{N}} (a_r - a_{r+\delta}) \, |r + \delta\rangle \otimes U_\delta(r) \, |c\rangle$$

Spatial distance

Any jump $\delta = (\pm 1, \pm 1)$ changes the sum of the coordinates $(x + y)$ by either $+2, 0$, or -2. This means the parity of $(x + y)$ is strictly conserved by the walk operator W_ϵ. Consequently, we split the lattice into two independent sublattices (even and odd). This gives two cases:

Case 1: r_1 and $r_2 \in$ different sublattices: If $(x_1 + y_1)$ and $(x_2 + y_2)$ have different parities, there exists no path connecting them using jumps from $\mathcal{N}$. The supremum is thus unbounded:

$$d_{W_\epsilon}(r_1, r_2) = \infty$$

Case 2: r_1 and $r_2 \in$ the same sublattice: If $(x_1 + y_1)$ and $(x_2 + y_2)$ have the same parity, a path always exists. The minimum number of steps to connect them is given by the maximum of the coordinate differences.

$$d_{W_\epsilon}(r_1, r_2) = \epsilon \max(|x_1 - x_2|, |y_1 - y_2|) = \epsilon \|r_1 - r_2\|_\infty$$

Lower Bound: Let $a(x, y) = \epsilon x$. Recall $W_\epsilon = C_2 D_y S_y C_1 D_x S_x$. The test operator a commutes with all operators except the shift S_x. Based on the shift definitions, S_x shifts the spatial support by $-e_x$ on the P_0 coin subspace and $+e_x$ on the P_1 subspace.

To find the operator $[S_x, x]$, we evaluate its action on a basis state $|x\rangle \otimes |c\rangle$:

$$\begin{aligned} [S_x, x]\big(|x\rangle \otimes |c\rangle\big) &= S_x\big(x\,|x\rangle \otimes |c\rangle\big) - x\big(S_x(|x\rangle \otimes |c\rangle)\big) \\ &= x\big(|x-1\rangle \otimes P_0\,|c\rangle + |x+1\rangle \otimes P_1\,|c\rangle\big) - x\big(|x-1\rangle \otimes P_0\,|c\rangle + |x+1\rangle \otimes P_1\,|c\rangle\big) \end{aligned}$$

Because the position operator x multiplies the state by its coordinate value, the second term evaluates as:

$$x\big(|x-1\rangle \otimes P_0\,|c\rangle + |x+1\rangle \otimes P_1\,|c\rangle\big) = (x-1)\,|x-1\rangle \otimes P_0\,|c\rangle + (x+1)\,|x+1\rangle \otimes P_1\,|c\rangle$$

Subtracting this from the first term gives:

$$\begin{aligned} [S_x, x]\big(|x\rangle \otimes |c\rangle\big) &= |x-1\rangle \otimes P_0\,|c\rangle - |x+1\rangle \otimes P_1\,|c\rangle \\ &= S_x(I \otimes \sigma_z)\big(|x\rangle \otimes |c\rangle\big) \end{aligned}$$

Therefore, $[S_x, x] = S_x(I \otimes \sigma_z)$. Substituting this back into the walk commutator:

$$[W_\epsilon, a] = \epsilon\, C_2 D_y S_y C_1 D_x [S_x, x] = \epsilon W_\epsilon (I \otimes \sigma_z)$$

and the operator norm is:

$$\|[W_\epsilon, a]\| = \epsilon \|W_\epsilon (I \otimes \sigma_z)\| = \epsilon \times 1 = \epsilon$$

This proves $a(x, y) = \epsilon x$ is a valid test function. By symmetry, $a(x, y) = \epsilon y$ is also valid. Evaluating these functions between r_1 and r_2 gives the differences $\epsilon|x_1 - x_2|$ and $\epsilon|y_1 - y_2|$. Taking the supremum:

$$d_{W_\epsilon}(r_1, r_2) \geq \epsilon \max(|x_1 - x_2|, |y_1 - y_2|) = \epsilon \|r_1 - r_2\|_\infty$$

Upper Bound: Let $a \in \mathcal{A}$ be any test function satisfying the norm constraint $\|[W_\epsilon, a]\| \leq \epsilon$. We define $\Delta a = W_\epsilon^\dagger a W_\epsilon - a$. Therefore, its norm:

$$\|\Delta a\| = \|W_\epsilon^\dagger a W_\epsilon - a\| = \|W_\epsilon^\dagger [a, W_\epsilon]\| = \|[W_\epsilon, a]\| \leq \epsilon$$

Next, we define the time evolved observable after N steps as $a(N) = (W_\epsilon^\dagger)^N a W_\epsilon^N$. We can expand the total change over N steps as a telescoping sum:

$$a(N) - a(0) = \sum_{t=0}^{N-1} (W_\epsilon^\dagger)^t (\Delta a) W_\epsilon^t$$

Taking the operator norm and applying the triangular inequality gives:

$$\|a(N) - a(0)\| \leq \sum_{t=0}^{N-1} \|(W_\epsilon^\dagger)^t (\Delta a) W_\epsilon^t\| = \sum_{t=0}^{N-1} \|\Delta a\| \leq N\epsilon$$

We now apply this global operator bound to our specific spatial points. Let $N = \|r_1 - r_2\|_\infty$. Assume that the function is shifted such that $a_{r_1} = 0$. We evaluate the expectation value of the evolved operator $a(N)$ on a state localized at the origin of our path, $|\psi_1\rangle = |r_1\rangle \otimes |c\rangle$:

$$|\langle\psi_1| a(N) - a(0) |\psi_1\rangle| \leq N\epsilon$$

Since $\langle\psi_1| a(0) |\psi_1\rangle = a_{r_1} = 0$, we have:

$$|\langle\psi_1| (W_\epsilon^\dagger)^N a W_\epsilon^N |\psi_1\rangle| = |\langle\phi_N| a |\phi_N\rangle| \leq N\epsilon$$

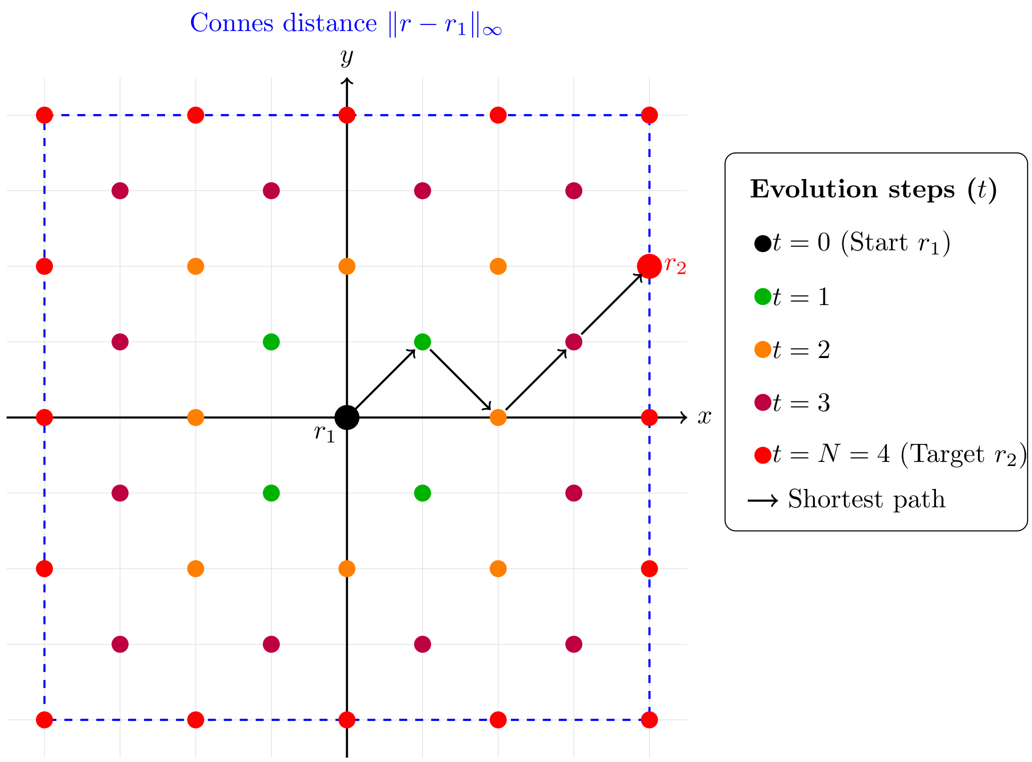
where $|\phi_N\rangle = W_\epsilon^N |\psi_1\rangle$ is the state evolved after N steps.

By the Causality theorem proven previously, the spatial support of $|\phi_N\rangle$ is contained within the causal light cone of radius N centered at r_1. Since $N = \|r_1 - r_2\|_\infty$, the target site r_2 is exactly on the boundary of this light cone. Because the expectation value $\langle\phi_N| a |\phi_N\rangle$ over the entire causal cone is bounded by $N\epsilon$, the test function a cannot physically accumulate a localized spatial difference greater than this Lieb-Robinson bound without violating causality. Therefore:

$$|a_{r_1} - a_{r_2}| \leq N\epsilon = \epsilon\|r_1 - r_2\|_\infty$$

By combining the lower bound and the upper bound, the Connes distance for points on the same sublattice is:

$$d_{W_\epsilon}(r_1, r_2) = \epsilon\|r_1 - r_2\|_\infty$$



5.2.2 Noncommutative Algebra

We evaluate the Connes distance for the quantum walk on the discrete lattice $\mathbb{Z}^2$ while keeping the full noncommutative algebra of bounded operators $\mathcal{A} = \mathcal{B}(\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2)$ (unlike the commutative case above).
This algebra is generated by

$$a = V_k S_q \otimes I_2$$

where

$$S_q |r\rangle = |r + q\rangle$$
$$V_k |r\rangle = e^{ik\cdot r} |r\rangle$$

In order to evaluate the commutator $[W_\epsilon, a]$, we study the action of W_ϵ on a basis state $|r\rangle \otimes |c\rangle$. As shown above (in the causality section), the walk applies a superposition of jumps $\delta \in \mathcal{N} = \{(\pm 1, \pm 1)\}$, and coin/gauge matrices $U_\delta(r)$:

$$W_\epsilon\big(|r\rangle \otimes |c\rangle \big) = \sum_{\delta \in \mathcal{N}} |r + \delta\rangle \otimes U_\delta(r) |c\rangle$$

Recall (from causality) that $U_\delta(r)$ depend on the position because they contain the magnetic gauge phases D_x and D_y.

First

$$a\big(\,|r\rangle\otimes|c\rangle\,\big)=e^{ik\cdot r}\,|r+q\rangle\otimes|c\rangle$$
$$W_\epsilon a\big(\,|r\rangle\otimes|c\rangle\,\big)=e^{ik\cdot r}\sum_{\delta\in\mathcal{N}}|r+q+\delta\rangle\otimes U_\delta(r+q)\,|c\rangle$$

Then

$$W_\epsilon\big(\,|r\rangle\otimes|c\rangle\,\big)=\sum_{\delta\in\mathcal{N}}|r+\delta\rangle\otimes U_\delta(r)\,|c\rangle$$
$$aW_\epsilon\big(\,|r\rangle\otimes|c\rangle\,\big)=\sum_{\delta\in\mathcal{N}}e^{ik\cdot(r+\delta)}\,|r+\delta+q\rangle\otimes U_\delta(r)\,|c\rangle$$

Therefore, subtracting these two results:

$$[W_\epsilon,a]\big(\,|r\rangle\otimes|c\rangle\,\big)=e^{ik\cdot r}\sum_{\delta\in\mathcal{N}}|r+q+\delta\rangle\otimes\Big(U_\delta(r+q)-e^{ik\cdot\delta}U_\delta(r)\Big)\,|c\rangle$$

Let $K_\delta(r)=U_\delta(r+q)-e^{ik\cdot\delta}U_\delta(r)$. Consequently, the norm of the commutator becomes:

$$\|[W_\epsilon,a]\|^2=\|\sum_{\delta_1\delta_2\in\mathcal{N}}\langle r+q+\delta_1|r+q+\delta_2\rangle\otimes\langle c|\,K_{\delta_1}(r)^\dagger K_{\delta_2}(r)\,|c\rangle\,\|$$

Since $\delta\in\{(1,1),(1,-1),(-1,1),(-1,-1)\}$, $\langle r+q+\delta_1|r+q+\delta_2\rangle=0$ unless $\delta_1=\delta_2$.
Therefore, the condition $\|[W_\epsilon,a]\|\le\epsilon$ becomes:

$$\begin{aligned}\|[W_\epsilon,a]\|&=\|\sum_{\delta\in\mathcal{N}}K_\delta(r)^\dagger K_\delta(r)\|^{\frac{1}{2}}=\sup_{r\in\mathbb{Z}^2}\|\sum_{\delta\in\mathcal{N}}K_\delta(r)^\dagger K_\delta(r)\|^{\frac{1}{2}}\\&=\sup_{r\in\mathbb{Z}^2}\|\sum_{\delta\in\mathcal{N}}\big(U_\delta(r+q)-e^{ik\cdot\delta}U_\delta(r)\big)^\dagger\big(U_\delta(r+q)-e^{ik\cdot\delta}U_\delta(r)\big)\|^{\frac{1}{2}}\\&\le\epsilon\end{aligned}$$

The action of W_ϵ to express U_δ: Recall the definitions from the causality section. The walk operator is $W_\epsilon=C_2D_yS_yC_1D_xS_x$.
We defined $s_x,s_y\in\{-1,1\}$ as the direction of the discrete shifts along the x and y axes and $P_{-1}=P_0=|0\rangle\langle 0|$ and $P_1=|1\rangle\langle 1|$ as the coin projectors in the jump directions. Thus, a single jump is $\delta=s_xe_x+s_ye_y$.
Applying the walk operator to a state $|r\rangle\otimes|c\rangle$:

1. First Shift (S_x): The walker shifts to $r+s_xe_x$, conditioned by P_{s_x}.
2. First Gauge Phase (D_x): The walker picks up the phase at its new position, applying the diagonal unitary $e^{i\epsilon A_x(r+s_xe_x)\sigma_z}$.
3. First Coin Mixing (C_1): The coin state is mixed by applying H.

4. Second Shift (S_y): The walker shifts along the y axis to the final position $r + s_x e_x + s_y e_y = r + \delta$, conditioned by P_{s_y}.
5. Second Gauge Phase (D_y): The walker picks up the phase at its final position, applying $e^{i\epsilon A_y(r+\delta)\sigma_z}$.
6. Second Coin Mixing (C_2): $M_\epsilon H$ is applied.

We obtain

$$U_\delta(r) = (M_\epsilon H)\, e^{i\epsilon A_y(r+\delta)\sigma_z} P_{s_y}\, H\, e^{i\epsilon A_x(r+s_x e_x)\sigma_z} P_{s_x}$$

No magnetic field: In the absence of the magnetic field, $A_x = A_y = 0$, which implies $U_\delta(r) = M_\epsilon H P_{s_y} H P_{s_x}$. In that case, U_δ is independent from the position r, and because $U_\delta(r+q) = U_\delta(r)$, $K_\delta(r) = U_\delta(r)(1 - e^{ik\cdot\delta})$.
The translation parameter q vanishes, reducing the bound to a constraint on k.

Non zero magnetic field: $U_\delta(r)$ depend on the position r in that case, which implies $U_\delta(r+q) \neq U_\delta(r)$
To keep the commutator norm lower than ϵ, any large shift q must be balanced by choosing an appropriate k because the $e^{ik\cdot\delta}$ is forced to compensate for the gauge difference between r and $r+q$. Therefore, the magnetic field entagles the position and the momentum.

Numerical Approach

Let the initial state be $|\psi_0\rangle$ and the evolved one $|\psi_t\rangle = W_t\,|\psi_0\rangle$ with $W_t = (W_\epsilon)^N$. Their associated state functionals are respectively

$$\begin{aligned} C_{\psi_0}(k,q) &= \langle\psi_0|a|\psi_0\rangle = \langle\psi_0|V_k S_q \otimes I_c|\psi_0\rangle \\ C_{\psi_t}(k,q) &= \langle\psi_t|a|\psi_t\rangle = \langle\psi_0|{W_t}^\dagger\, a\, W_t|\psi_0\rangle \\ &= \langle\psi_0|{W_t}^\dagger\,(V_k S_q \otimes I_c)\, W_t|\psi_0\rangle \end{aligned}$$

Therefore, the Connes distance is given by

$$\begin{aligned} d_{W_\epsilon}(\psi_t,\psi_0) &= \sup_{a\in\mathcal{A}} \Big|\, \langle\psi_t|a|\psi_t\rangle - \langle\psi_0|a|\psi_0\rangle \,\Big| \\ &= \sup_{k,q} \Big|\, \langle\psi_0|{W_t}^\dagger\,(V_k S_q \otimes I_c)\, W_t - (V_k S_q \otimes I_c)|\psi_0\rangle \,\Big| \\ &= \max_{k\in[-\pi,\pi[^2,\; q\in\mathbb{Z}^2} \Big| C_{\psi_t}(k,q) - C_{\psi_0}(k,q) \Big| \end{aligned}$$

subject to the constraint

$$\|[W_\epsilon, a]\| = \max_{r\in\text{grid}} \left[\lambda_{\max} \sum_{\delta\in\mathcal{N}} \big(U_\delta(r+q) - e^{ik\cdot\delta} U_\delta(r)\big)^\dagger \big(U_\delta(r+q) - e^{ik\cdot\delta} U_\delta(r)\big) \right]^{\frac{1}{2}} \leq \epsilon$$

6 Conclusion

We established a framework for studying the Dirac quantum walk through the lens of Noncommutative Geometry. By discretizing the Dirac Hamiltonian, we constructed a unitary evolution operator and mathematically proved that this DTQW preserves relativistic causality. We demonstrated that the introduction of a magnetic gauge field fundamentally changes the structure of the lattice preventing the spatial translation operators from commuting and entangling the momentum and spatial coordinates. Using Connes distance, we successfully recovered the classical Euclidean metric in the commutative limits and highlighted the metric changes that appear when the algebra becomes noncommutative. Unlike the continuous derivations that gave the exact analytical solutions, the discrete noncommutative framework introduces highly complex constraints. Consequently, we have formulated the Connes distance in this regime as a constrained optimization problem. We are currently working on estimating these distances numerically.

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